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Comments on gauge equivalence in noncommutative geometry

Tsuguhiko Asakawa and Isao Kishimoto

Department of Physics, Kyoto University
Kyoto 606-8502, Japan
E-mail: asakawa@gauge.scphys.kyoto-u.ac.jp,
ikishimo@gauge.scphys.kyoto-u.ac.jp

Abstract: We investigate the transformation from ordinary gauge field to noncommutative one which was introduced by N. Seiberg and E. Witten (hep-th/9908142). It is shown that the general transformation which is determined only by gauge equivalence has a path dependence in 'θ-space'. This ambiguity is negligible when we compare the ordinary Dirac-Born-Infeld action with the noncommutative one in the U(1) case, because of the U(1) nature and slowly varying field approximation. However, in general, in the higher derivative approximation or in the U(N) case, the ambiguity cannot be neglected due to its noncommutative structure. This ambiguity corresponds to the degrees of freedom of field redefinition.

Keywords: Brane Dynamics in Gauge Theories, Gauge Symmetry.

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Contents

1. Introduction

Gauge theories on noncommutative spaces have been investigated for many years from mathematical and physical viewpoint ([2] and the references in [1]). Especially in string theory, the worldvolume theory of D-branes in a background B-field is described by noncommutative Yang-Mills or Dirac-Born-Infeld theory.

Recently, Seiberg and Witten [1] argued the equivalence between ordinary gauge field theory and the noncommutative one as the low energy effective theories of open strings: they arise from the same two-dimensional field theory regularlized in different ways, so that there must be a transformation among them. In [1], this transformation is uniquely given by the gauge equivalence relation, and this implies the equivalence between ordinary Dirac-Born-Infeld action and the noncommutative one.

In this short note, we re-examine the validity of above arguments and point out that the transformation of [1] has in general ambiguities. In section 2, we begin with the gauge equivalence relation between two nearby points in the 'θ-space' and show that there is ambiguity with arbitrary constant parameters. Then, we discuss the path dependence of the transformation in the 'θ-space', which is found by considering the commutator of two transformations. This implies the existence of another ambiguity. In section 3 we investigate these ambiguities from different viewpoint. Next in section 4, we consider the U(1) case in slowly varying field approximation. This is the situation of [1] in comparing the ordinary Dirac-Born-Infeld action with the noncommutative one. In this case the ambiguities do not affect the result of [1], because of the U(1) nature and of neglecting the higher derivative terms. In section 5, we summarize the paper and give some discussions. In the note added, we argue that the path dependence can be reduced to the field redefinition.

2. Gauge equivalence relation

In [1], they obtained a transformation from ordinary gauge field A_i (and gauge parameter λ) to noncommutative gauge field $\hat{A}_i$ (and gauge parameter $\hat{\lambda}$) by demanding the gauge equivalence relation between them. However, we show that their statement has generally ambiguities. Here we investigate the gauge equivalence relation carefully.

Consider a noncommutative, associative algebra denoted by $\mathcal{A}_\theta = (\mathfrak{g} \otimes C^\infty, *)$, where $\mathfrak{g}$ is some Lie algebra and the $*$ product is defined to be the tensor product of matrix multiplication with the product of functions such as $f(x) * g(x) := \exp(i\frac{\theta^{kl}}{2}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^l}) f(y)g(z)|_{y=z=x}$ with constant antisymmetric tensor $\theta^{kl} = -\theta^{lk}$. Note that θ's are arbitrary parameters. We denote this parameter space of the whole set of algebra $\{\mathcal{A}_\theta\}_{\theta\in\vartheta}$ as θ-space ϑ.

We assume there exists some sort of mapping from $\mathcal{A}_\theta$ to another $\mathcal{A}_{\tilde{\theta}}$ in a way that preserve gauge equivalence relation which is described by the following equation in terms of gauge fields and gauge parameters $\tilde{A}_i(\hat{A}), \tilde{\lambda}(\hat{A}, \hat{\lambda}) \in \mathcal{A}_{\tilde{\theta}}$ and $\hat{A}_i, \hat{\lambda} \in \mathcal{A}_\theta$:

$$\tilde{A}_i(\hat{A}) + \tilde{\delta}_{\tilde{\lambda}} \tilde{A}_i(\hat{A}) = \tilde{A}_i(\hat{A} + \hat{\delta}_{\hat{\lambda}} \hat{A}) \,, \tag{2.1}$$

where $\hat{\delta}_{\hat{\lambda}}$ is the gauge transformation with infinitesimal $\hat{\lambda}$, i.e. $\hat{\delta}_{\hat{\lambda}} \hat{A}_i = \hat{D}_i \hat{\lambda} = \partial_i \hat{\lambda} - i[\hat{A}_i, \hat{\lambda}]$.[1] and likewise for $\tilde{\delta}_{\tilde{\lambda}}$. This relation means that the diagram below is commutative.

$$\begin{tikzcd}
\hat{A}'_i \arrow[r] & \tilde{A}'_i \\
\hat{A}_i \arrow[u, "\hat{\delta}_{\hat{\lambda}}"] \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & \tilde{A}_i \arrow[u, "\tilde{\delta}_{\tilde{\lambda}}"']
\end{tikzcd}$$

Especially in the case of nearby points in ϑ, i.e. $\tilde{\theta} = \theta + \delta\theta$ with infinitesimal $\delta\theta$, eq. (2.1) is written in the variational form as

$$\hat{\delta}_{\hat{\lambda}} \delta \hat{A}_i = \delta \hat{\delta}_{\hat{\lambda}} \hat{A}_i \tag{2.2}$$

by writing $\tilde{A} = \hat{A} + \delta\hat{A}(\hat{A}) + \mathcal{O}(\delta\theta^2)$, $\tilde{\lambda} = \hat{\lambda} + \delta\hat{\lambda}(\hat{A}, \hat{\lambda}) + \mathcal{O}(\delta\theta^2)$ and expanding (2.1) to the first order in $\delta\theta$.

[1] In this paper, $[\hat{A}, \hat{B}] = \hat{A} * \hat{B} - \hat{B} * \hat{A}$, $\{\hat{A}, \hat{B}\} = \hat{A} * \hat{B} + \hat{B} * \hat{A}$.

We first look for the solution of (2.1) by using the method described in the next section. Eq. (2.1) can be easily rewritten as[2]

$$\hat{\delta}_{\hat{\lambda}}\delta\hat{A}_i - \hat{D}_i\delta\hat{\lambda} + i[\delta\hat{A}_i, \hat{\lambda}] = -\frac{1}{2}\delta\theta^{kl}\{\partial_k\hat{A}_i, \partial_l\hat{\lambda}\}, \tag{2.3}$$

which corresponds to the $n = 1$ case of (3.2). Note that this form is actually the same one as given in [1]: the $\delta\theta$ version of [1, eq. (3.4)]. It is solved most generally by (see next section for detail)

$$\begin{aligned}
\delta\hat{A}_i &= -\frac{1}{4}\delta\theta^{kl}\{\hat{A}_k, \partial_l\hat{A}_i + \hat{F}_{li}\} + \alpha\delta\theta^{kl}\hat{D}_i\hat{F}_{kl} + \beta\delta\theta^{kl}\hat{D}_i[\hat{A}_k, \hat{A}_l]\,, \\
\delta\hat{\lambda} &= \frac{1}{4}\delta\theta^{kl}\{\partial_k\hat{\lambda}, \hat{A}_l\} + 2\beta\delta\theta^{kl}[\partial_k\hat{\lambda}, \hat{A}_l]\,, \\
\delta\hat{F}_{ij} &= \frac{1}{4}\delta\theta^{kl}\left(2\{\hat{F}_{ik}, \hat{F}_{jl}\} - \{\hat{A}_k, \hat{D}_l\hat{F}_{ij} + \partial_l\hat{F}_{ij}\}\right) - \\
&\quad -i\alpha\delta\theta^{kl}[\hat{F}_{ij}, \hat{F}_{kl}] - i\beta\delta\theta^{kl}[\hat{F}_{ij}, [\hat{A}_k, \hat{A}_l]]\,,
\end{aligned} \tag{2.4}$$

where $\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i[\hat{A}_i, \hat{A}_j]$ is the field strength and α, β are arbitrary constants. ($\alpha = \beta = 0$ case corresponds to [1, eq. (3.8)].) The presence of arbitrary parameters α and β implies that, with the requirement of the gauge equivalence alone, there exists in general ambiguity in determining an infinitesimal mapping. However, note that this ambiguity has rather trivial origin because we look for two functions $\delta\hat{A}_i$, $\delta\hat{\lambda}$ as the solution of one equation (2.2), and that the terms with α, β have formally a form of some gauge transformation. Recall that the mapping that satisfies the gauge equivalence relation is the one which maps gauge orbits from $\mathcal{A}_\theta$ to $\mathcal{A}_{\tilde{\theta}}$ rather than gauge fields themselves. Therefore, this kind of ambiguity is not relevant when we discuss only gauge equivalence classes.

However, applying $\delta\theta$-variation twice, we will encounter the second kind of ambiguities. Denote each variation as δ_1 and δ_2, respectively, which are in general different direction with each other in the θ-space, and consider their commutation relation acting on $\hat{A}_i$:

$$[\delta_1, \delta_2]\hat{A}_i\,, \tag{2.5}$$

[2]Use the following relations

$$\begin{aligned}
\delta\{f, g\} &= \{\delta f, g\} + \{f, \delta g\} + \frac{i}{2}\delta\theta^{pq}[\partial_p f, \partial_q g]\,, \\
\delta\hat{D}_i f &= \hat{D}_i\delta f - i[\delta\hat{A}_i, f] + \frac{i}{2}\delta\theta^{pq}\{\partial_p\hat{A}_i, \partial_q f\}.
\end{aligned}$$

which measures the 'path dependence' in θ-space ϑ. Using the transformation (2.4) twice, we obtain explicitly

$$
\begin{aligned}
[\delta_1, \delta_2]\hat{A}_i = {} & \delta_1 \left(-\frac{1}{4}\delta\theta_2^{kl}\{\hat{A}_k, \partial_l \hat{A}_i + \hat{F}_{li}\} + \alpha\delta\theta_2^{kl}\hat{D}_i\hat{F}_{kl} + \beta\delta\theta_2^{kl}\hat{D}_i[\hat{A}_k, \hat{A}_l] \right) - (1 \leftrightarrow 2) \\
= {} & \frac{1}{16}\delta\theta_2^{kl}\delta\theta_1^{pq}\Big(4i[\hat{F}_{kp}, \partial_l\partial_q\hat{A}_i] + 4[\hat{F}_{kp}, [\hat{A}_l, \partial_q\hat{A}_i] + [\hat{A}_q, \partial_l\hat{A}_i]] - \\
& \qquad - [\partial_k\hat{A}_p + \hat{F}_{kp}, [\hat{A}_l, \hat{D}_i\hat{A}_q]] + [\partial_p\hat{A}_k + \hat{F}_{pk}, [\hat{A}_q, \hat{D}_i\hat{A}_l]] + \\
& \qquad + \{\hat{A}_k, \{\hat{F}_{lq}, \hat{D}_i\hat{A}_p\}\} - \{\hat{A}_p, \{\hat{F}_{ql}, \hat{D}_i\hat{A}_k\}\} - \\
& \qquad - i\{\hat{A}_p, \{\hat{A}_k, [\hat{A}_l, \hat{D}_i\hat{A}_q]\}\} + i\{\hat{A}_k, \{\hat{A}_p, [\hat{A}_q, \hat{D}_i\hat{A}_l]\}\} + \\
& \qquad + 2i[\partial_p\hat{A}_k, \hat{D}_i\partial_q\hat{A}_l] - 2i[\partial_k\hat{A}_p, \hat{D}_i\partial_l\hat{A}_q] - \\
& \qquad - [[\hat{A}_k, \hat{A}_p], \hat{D}_i\partial_q\hat{A}_l] + [[\hat{A}_p, \hat{A}_k], \hat{D}_i\partial_l\hat{A}_q] - \\
& \qquad - \{\hat{A}_k, \{\hat{A}_p, \hat{D}_i\partial_q\hat{A}_l\}\} + \{\hat{A}_p, \{\hat{A}_k, \hat{D}_i\partial_l\hat{A}_q\}\}\Big) + \\
& + \hat{D}_i\Big(\delta\theta_2^{kl}\delta\theta_1^{pq}\Big(i\alpha^2[\hat{F}_{kl}, \hat{F}_{pq}] + i\beta^2[[\hat{A}_k, \hat{A}_l], [\hat{A}_p, \hat{A}_q]] + \\
& \qquad + i\alpha\beta([[\hat{A}_k, \hat{A}_l], \hat{F}_{pq}] - [[\hat{A}_p, \hat{A}_q], \hat{F}_{kl}]) + \\
& \qquad + \frac{1}{4}\alpha(\{\partial_k\hat{F}_{pq}, \hat{A}_l\} - \{\partial_p\hat{F}_{kl}, \hat{A}_q\}) + \\
& \qquad + \frac{1}{4}\beta(\{\partial_k[\hat{A}_p, \hat{A}_q], \hat{A}_l\} - \{\partial_p[\hat{A}_k, \hat{A}_l], \hat{A}_q\})\Big) + \\
& \quad + \delta\theta_2^{kl}\delta_1(\alpha\hat{F}_{kl} + \beta[\hat{A}_k, \hat{A}_l]) - \delta\theta_1^{pq}\delta_2(\alpha\hat{F}_{pq} + \beta[\hat{A}_p, \hat{A}_q])\Big) . \qquad (2.6)
\end{aligned}
$$

Note that the sum of all α, β dependent terms again has the form of some gauge transformation (with $\hat{A}_i$ dependent parameter). This is easily understood by noticing that the gauge transformations are closed under commutation relations and the requirement (2.2). Contrary, the α, β independent terms are nontrivial and they do not vanish in general. That is, there exists path dependence if we repeat variations more than one step in $\delta\theta$. In terms of the gauge equivalence, (2.6) means the following. In the same sense as we discussed below (2.4) for the one-step variation, a gauge orbit in $\mathcal{A}_\theta$ is mapped to an orbit in $\mathcal{A}_{\theta+\delta\theta_1+\delta\theta_2}$, but now depending on the path: orbits mapped along two paths are not the same.

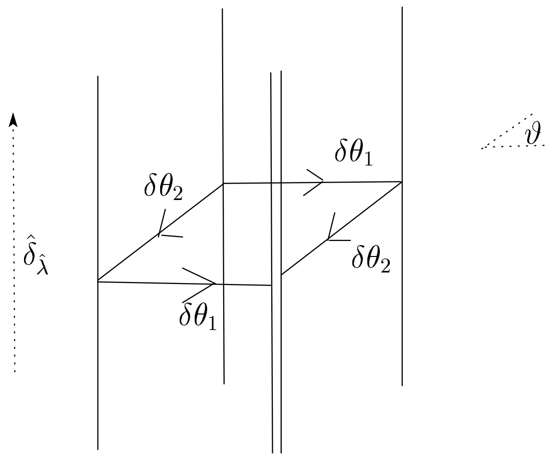

Figure 1: A vertical line denotes gauge orbit on a point in θ-space. The double line denotes two different orbits on the same point.

This second type of ambiguities accumulates globally in θ-space, if we consider any mapping from $\mathcal{A}_\theta$ to $\mathcal{A}_{\tilde\theta}$ at a finite distance apart in θ-space. Transformation on gauge fields is given by the integration over $\delta\theta$ by specifying a path by hand as

$$\tilde{A} = \int_{\text{path}} \delta\hat{A}\,. \tag{2.7}$$

Of course, $\delta\hat{A}$ suffers also from the first type of ambiguities. If we further fix α and β by hand, i.e. select a representative, then $\tilde{A}$ is uniquely 'determined'. The procedure described in [1], where the functional $\hat{A}(A)$ is determined order by order in θ, is exactly the one discussed here. In fact, the solution of [1] corresponds to taking $\alpha = \beta = 0$ and the 'straight line' in θ-space as the path of integration. Here the 'straight line' corresponds to the formal exponentiation of the infinitesimal transformation (2.4).

Note that there exists no rule to select a particular path from the standpoint of gauge theory (or more precisely a space of the whole set of algebra $\{\mathcal{A}_\theta\}_{\theta\in\vartheta}$.) We need some physical requirement. In section 4 we discuss the equivalence of actions between ordinary gauge theory and noncommutative one in this point of view.

3. More comments on ambiguity

In this section, we investigate the gauge equivalence relation (2.1) from another viewpoint.

To get a solution of (2.1) directly, we expand formally $\tilde{A}_i$ as the power series in $\delta\theta = \tilde\theta - \theta$:

$$\tilde{A}_i = \sum_{n=0}^{\infty} \hat{A}_i^{(n)}\,, \qquad \tilde\lambda = \sum_{n=0}^{\infty} \hat\lambda^{(n)}\,, \tag{3.1}$$

where $\hat{A}_i^{(n)}, \hat\lambda^{(n)} \in \mathcal{A}_\theta$ are of $\mathcal{O}(\delta\theta^n)$, and $\hat{A}_i^{(0)} = \hat{A}_i$, $\hat\lambda^{(0)} = \hat\lambda$. Substituting this formal expansion (3.1) into (2.1), the equation of $\mathcal{O}(\delta\theta^n)$ is

$$\begin{aligned} &\hat\delta_{\hat\lambda}\hat{A}_i^{(n)} - \hat{D}_i\hat\lambda^{(n)} + i\left[\hat{A}_i^{(n)}, \hat\lambda\right] = \\ &\qquad = -i\sum \left(\frac{i}{2}\right)^r \delta\theta^{k_1 l_1}\cdots\delta\theta^{k_r l_r}\left[\partial_{k_1}\cdots\partial_{k_r}\hat{A}_i^{(p)}, \partial_{l_1}\cdots\partial_{l_r}\hat\lambda^{(q)}\right\}, \end{aligned} \tag{3.2}$$

where the summation ranges in $p+q+r = n,\ p,q,r \geq 0,\ p \neq n,\ q \neq n$, and $[\ ,\ \}$ denotes the anti-commutator $\{\ ,\ \}$ (the commutator $[\ ,\ ]$) if r is odd (even). This equation implies that $\hat{A}_i^{(n)}, \hat\lambda^{(n)}$ on the left hand side are determined by $\mathcal{O}(\delta\theta^{n-1})$ quantities on the right hand side.

Concrete procedure to get $\hat{A}_i^{(n)}$, $\hat\lambda^{(n)}$ is as follows: substitute the solution $\hat{A}_i^{(k)}$, $\hat\lambda^{(k)}$, $(k = 1, \ldots, n-1)$ of (3.2) to the right hand side, express $\hat{A}^{(n)}$, $\hat\lambda^{(n)}$ as a polynomial of $\delta\theta^n, \hat{A}_i, \partial_j\hat{A}_k, \ldots, \hat\lambda, \partial_l\hat\lambda, \ldots$ in the most general form with suitable indices[3] and substitute it to the left hand side of (3.2), then we can determine the coefficients in the polynomial.

[3] We assume here that a transformation from $\hat{A}_i, \hat\lambda$ to $\tilde{A}_i, \tilde\lambda$ can be expressed by some polynomial of $\hat{A}_i, \hat\lambda, \partial_j\hat{A}_k, \ldots, \delta\theta^{mn}$ alone and indices are contracted among them.

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However, suppose there exist some functions $\hat{A}_i^{0(n)}, \hat{\lambda}^{0(n)}$ such that

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_i^{0(n)} - \hat{D}_i \hat{\lambda}^{0(n)} + i[\hat{A}_i^{0(n)}, \hat{\lambda}] = 0 \,. \tag{3.3}$$

Then $\hat{A}_i^{(n)} + \hat{A}_i^{0(n)}, \hat{\lambda}^{(n)} + \hat{\lambda}^{0(n)}$ are also a solution of (3.2) if $\hat{A}_i^{(n)}, \hat{\lambda}^{(n)}$ satisfy (3.2). In fact, we can construct such $\hat{A}_i^{0(n)}, \hat{\lambda}^{0(n)}$ as follows.

Noting that

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_i = \partial_i \hat{\lambda} - i[\hat{A}_i, \hat{\lambda}] \,, \qquad \hat{\delta}_{\hat{\lambda}} \hat{D}_i \hat{A}_j = \hat{D}_i(\partial_j \hat{\lambda}) - i[\hat{D}_i \hat{A}_j, \hat{\lambda}], \ldots \,, \tag{3.4}$$

and that $\hat{\delta}_{\hat{\lambda}}$ and the commutator [,] satisfy Leibnitz rule, we obtain the following identity for any polynomial $\hat{G}$ of $\hat{A}_i, \hat{D}_i \hat{A}_j, \ldots$ in $\mathcal{A}_\theta$:

$$\hat{\delta}_{\hat{\lambda}} \hat{G}(\hat{A}_j, \hat{D}_k \hat{A}_l, \ldots) - \hat{\delta}'_{\hat{\lambda}} \hat{G}(\hat{A}_j, \hat{D}_k \hat{A}_l, \ldots) + i \left[\hat{G}(\hat{A}_j, \hat{D}_k \hat{A}_l, \ldots), \hat{\lambda} \right] = 0 \,, \tag{3.5}$$

where $\hat{\delta}'_{\hat{\lambda}}$ acts like $\partial_i \hat{\lambda} \cdot \frac{\delta}{\delta \hat{A}_i}$ i.e., replaces $\hat{A}_i$ with $\partial_i \hat{\lambda}$ but does not act on $\hat{A}_i$ in $\hat{D}_i$ (hence $\hat{D}_i \hat{\delta}'_{\hat{\lambda}} = \hat{\delta}'_{\hat{\lambda}} \hat{D}_i$). In the same way,

$$\hat{\delta}_{\hat{\lambda}} \hat{F}_{ij} = -i[\hat{F}_{ij}, \hat{\lambda}] \,, \qquad \hat{\delta}_{\hat{\lambda}} \hat{D}_k \hat{F}_{ij} = -i[\hat{D}_k \hat{F}_{ij}, \hat{\lambda}], \ldots \,, \tag{3.6}$$

lead to

$$\hat{\delta}_{\hat{\lambda}} \hat{G}^F(\hat{F}_{jk}, \hat{D}_l \hat{F}_{mn}, \ldots) + i \left[\hat{G}^F(\hat{F}_{jk}, \hat{D}_l \hat{F}_{mn}, \ldots), \hat{\lambda} \right] = 0 \,, \tag{3.7}$$

where $\hat{G}^F(\hat{F}_{jk}, \hat{D}_l \hat{F}_{mn}, \ldots)$ is a polynomial of $\hat{F}_{jk}, \hat{D}_l \hat{F}_{mn}, \ldots$ in $\mathcal{A}_\theta$.

From (3.5) and (3.7), we get one type of solution of (3.3)

$$\begin{aligned} \hat{A}_i^{0(n)} &= \hat{G}_i^{(n)F}(\hat{F}_{jk}, \hat{D}_l \hat{F}_{mn}, \ldots ; \delta\theta^n) + \hat{D}_i \hat{G}^{(n)}(\hat{A}_j, \hat{D}_k \hat{A}_l, \ldots ; \delta\theta^n) \,, \\ \hat{\lambda}^{0(n)} &= \hat{\delta}'_{\hat{\lambda}} \hat{G}^{(n)}(\hat{A}_j, \hat{D}_k \hat{A}_l, \ldots ; \delta\theta^n) \,, \end{aligned} \tag{3.8}$$

This means that there is large ambiguity due to arbitrary polynomials $\hat{G}^{(n)}(\hat{A}_j, \hat{D}_k \hat{A}_l, \ldots ; \delta\theta^n)$, $\hat{G}_i^{(n)F}(\hat{F}_{jk}, \hat{D}_l \hat{F}_{mn}, \ldots ; \delta\theta^n)$ in each order in (3.1). This result is consistent with the ambiguity due to the path dependence of section 2.

In particular, if we take $\delta\theta$ infinitesimal, then the ambiguity is of the form

$$\begin{aligned} \hat{G}^{(1)}(\hat{A}_j, \hat{D}_k \hat{A}_l, \ldots ; \delta\theta^1) &= \beta_1 \delta\theta^{kl} [\hat{A}_k, \hat{A}_l] + \beta_2 \delta\theta^{kl} \hat{D}_k \hat{A}_l \,, \\ \hat{G}_i^{(1)F}(\hat{F}_{jk}, \hat{D}_l \hat{F}_{mn}, \ldots ; \delta\theta^1) &= \alpha_1 \delta\theta^{kl} \hat{D}_i \hat{F}_{kl} \,, \end{aligned} \tag{3.9}$$

where $\alpha_1, \beta_1, \beta_2$ are arbitrary constants. Substituting (3.9) into (3.8), we get

$$\begin{aligned} \hat{A}_i^{0(1)} &= \delta\theta^{kl} (\alpha_1 \hat{D}_i \hat{F}_{kl} + \beta_1 \hat{D}_i [\hat{A}_k, \hat{A}_l] + \beta_2 \hat{D}_i \hat{D}_k \hat{A}_l) \\ &= \left(\alpha_1 + \frac{1}{2} \beta_2 \right) \delta\theta^{kl} \hat{D}_i \hat{F}_{kl} + \left(\beta_1 - \frac{1}{2} i \beta_2 \right) \delta\theta^{kl} \hat{D}_i [\hat{A}_k, \hat{A}_l] \,, \\ \hat{\lambda}^{0(1)} &= \delta\theta^{kl} (2\beta_1 [\partial_k \hat{\lambda}, \hat{A}_l] + \beta_2 \hat{D}_k \partial_l \hat{\lambda}) = (2\beta_1 - i\beta_2) \delta\theta^{kl} [\partial_k \hat{\lambda}, \hat{A}_l] \,. \end{aligned} \tag{3.10}$$

By redefinition of coefficients, this is the α, β dependent term in (2.4).

4. U(1) case

In this section, we consider the case where the gauge group is U(1). We assume here that $\hat{F}_{ij}$ is slowly varying so that we can ignore $\mathcal{O}(\partial\hat{F})$. This approximation is adopted when we consider the Dirac-Born-Infeld action. Precisely, we regard $\hat{F} \sim \partial\hat{A}$ as $\mathcal{O}(1)$ and count the order by (the number of ∂)–(that of $\hat{A}$). Note that $\hat{D}_i = \partial_i + \theta^{jk}\partial_j\hat{A}_i\partial_k + \mathcal{O}(\partial^4\hat{F}\partial)$, $\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i + \theta^{kl}\partial_k\hat{A}_i\partial_l\hat{A}_j + \mathcal{O}(\partial^4\hat{F})$, and that α, β dependent terms in $\delta\hat{A}_i$ in (2.4) are of $\mathcal{O}(\partial^2\hat{F})$, and hence negligible.

Eq. (2.6) reduces in this approximation to

$$[\delta_1, \delta_2]\hat{A}_i = \frac{1}{4}\delta\theta_2^{kl}\delta\theta_1^{pq}\hat{D}_i(\hat{A}_k\hat{A}_p\hat{F}_{lq}) + \mathcal{O}(\hat{A}\partial^4\hat{F})\,, \tag{4.1}$$

and in the same way, we obtain[4]

$$\begin{aligned}
[\delta_1, \delta_2]\hat{F}_{ij} = \frac{1}{16}\delta\theta_2^{kl}\delta\theta_1^{pq}\Big(4\Big(&i[\hat{D}_p\hat{F}_{ik}, \hat{D}_q\hat{F}_{jl}] - i[\hat{D}_k\hat{F}_{ip}, \hat{D}_l\hat{F}_{jq}] + \\
&+ [[\hat{F}_{ik}, \hat{F}_{jp}] + [\hat{F}_{ip}, \hat{F}_{jk}], \hat{F}_{lq}]\Big) + \\
&+ 4(i[\hat{F}_{kp}, \partial_l\partial_q\hat{F}_{ij}] + [\hat{F}_{kp}, [\hat{A}_q, \partial_l\hat{F}_{ij}] + [\hat{A}_l, \partial_q\hat{F}_{ij}]]) + \\
&+ 2i[[\hat{A}_p, \hat{A}_k], [\partial_l\hat{A}_q, \hat{F}_{ij}] + [\partial_q\hat{A}_l, \hat{F}_{ij}]] + \\
&+ i[\partial_q\hat{A}_l + \hat{F}_{ql}, [\hat{A}_p, [\hat{A}_k, \hat{F}_{ij}]]] - i[\partial_l\hat{A}_q + \hat{F}_{lq}, [\hat{A}_k, [\hat{A}_p, \hat{F}_{ij}]]] + \\
&+ i\{\hat{A}_p, \{\hat{F}_{lq}, [\hat{A}_k, \hat{F}_{ij}]\}\} - i\{\hat{A}_k, \{\hat{F}_{ql}, [\hat{A}_p, \hat{F}_{ij}]\}\} - \\
&- \{\hat{A}_k, \{\hat{A}_p, [\hat{A}_q, [\hat{A}_l, \hat{F}_{ij}]]\}\} + \{\hat{A}_p, \{\hat{A}_k, [\hat{A}_l, [\hat{A}_q, \hat{F}_{ij}]]\}\} + \\
&+ i\{\hat{A}_k, \{\hat{A}_p, [\partial_l\hat{A}_q, \hat{F}_{ij}]\}\} - i\{A_p, \{\hat{A}_k, [\partial_q\hat{A}_l, \hat{F}_{ij}]\}\} - \\
&- 2[\partial_p\hat{A}_k, [\partial_q\hat{A}_l, \hat{F}_{ij}]] + 2[\partial_k\hat{A}_p, [\partial_l\hat{A}_q, \hat{F}_{ij}]]\Big) + \\
&+(\alpha, \beta \text{ dependent terms}) \\
= -\frac{1}{4}i\delta\theta_2^{kl}\delta\theta_1^{pq}[\hat{F}_{ij}, \hat{A}_k\hat{A}_p\hat{F}_{lq}] + \mathcal{O}(\partial^4\hat{F})\,.&
\end{aligned} \tag{4.2}$$

The right hand sides of (4.1) and (4.2) have terms that is not of $\mathcal{O}(\partial\hat{F})$ but is of the form of gauge transformation with gauge parameter $\frac{1}{4}\delta\theta_2^{kl}\delta\theta_1^{pq}\hat{A}_k\hat{A}_p\hat{F}_{lq}$.[5] This means that $\hat{A}_i$ and $\hat{F}_{ij}$ can be determined up to gauge transformation in such rough approximation of ignoring $\mathcal{O}(\partial\hat{F})$.

In [1] they showed that the ordinary Dirac-Born-Infeld lagrangean equals the noncommutative one up to total derivative terms and up to $\mathcal{O}(\partial\hat{F})$. They argued that the more general lagrangean

$$\hat{\mathcal{L}}_{\mathrm{DBI}} = \frac{1}{G_s}\sqrt{\det(G + \hat{F} + \Phi)} \tag{4.3}$$

[4]The first equality of (4.2) is exact even in the U(N) case and the second one is valid only in the U(1) case.

[5]Notice that, if $\theta^{kl} \neq 0$, $\hat{F}_{ij}$ is not gauge invariant even in the U(1) case.

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is invariant up to total derivative terms and up to $\mathcal{O}(\partial\hat{F})$ under the variation with respect to θ. The gauge field of ordinary Dirac-Born-Infeld theory is in $\mathcal{A}_{\theta=0}$ and noncommutative one is in $\mathcal{A}_{\theta\neq 0}$.[6] In their proof, eq. (2.4) with $\alpha=\beta=0$ is used. There is in general ambiguity due to α,β dependence in (2.4) but this is negligible.

As we discussed in the previous sections, there is ambiguity due to path dependence in θ-space. This implies that

$$\hat{\mathcal{L}}_{\mathrm{DBI}}|_{\tilde{\theta}} - \hat{\mathcal{L}}_{\mathrm{DBI}}|_{\theta} = \int_{\mathrm{path}} \delta\hat{\mathcal{L}}_{\mathrm{DBI}} \,. \tag{4.4}$$

However, this path dependence is in fact missing as seen from the gauge transformation form of (4.1) and (4.2). Therefore their proof of equivalence between the ordinary Dirac-Born-Infeld action and the noncommutative one (or more generally the equivalence of the action (4.3) in θ-space) is also justified in our context. This means that in this physical input (i.e. equivalence of the ordinary DBI action and noncommutative one) no ambiguity is restricted.

5. Conclusions and discussions

In this paper, we considered a transformation from $\hat{A}_i, \hat{\lambda} \in \mathcal{A}_\theta$ to $\tilde{A}_i, \tilde{\lambda} \in \mathcal{A}_{\tilde{\theta}}$ which is 'determined' by gauge equivalence. This transformation has large ambiguity due to path dependence in θ-space. However this ambiguity is negligible in particular in the U(1) case and in rough approximation of ignoring $\mathcal{O}(\partial\hat{F})$. This fact justifies the equivalence of noncommutative Dirac-Born-Infeld lagrangean (4.3) in the θ-space.

However the ambiguity is no longer negligible in the U(N) case or in the U(1) case if $\theta \neq 0$ and higher derivative correction is considered because the path dependence (2.6) is not of the form of gauge transformation. So if one considers higher derivative correction from the Dirac-Born-Infeld action or the U(N) generalization of (4.3) by using transformation determined only by gauge equivalence, we need a more careful argument. Geometrical interpretation of the variation with respect to θ such as (2.4) would be required.

Note added. eq. (2.6) can be rewritten as follows:[7]

[6] The antisymmetric tensor Φ is given by $\frac{1}{G+\Phi} = -\theta + \frac{1}{g+B}$, where G, g, B is the open string metric, the closed string metric and the NS 2-form field, respectively.

[7] eq. (4.2) can also be rewritten as follows:

$$\begin{aligned}
[\delta_1,\delta_2]\hat{F}_{ij} = \frac{1}{16}\delta\theta_2^{kl}\delta\theta_1^{pq}\Big(& 4(i[\hat{D}_p\hat{F}_{ik},\hat{D}_q\hat{F}_{jl}] - i[\hat{D}_k\hat{F}_{ip},\hat{D}_l\hat{F}_{jq}] + [[\hat{F}_{ik},\hat{F}_{jp}] + [\hat{F}_{ip},\hat{F}_{jk}],\hat{F}_{lq}]) + \\
& + 2i[\hat{F}_{kp},\hat{D}_l\hat{D}_q\hat{F}_{ij} + \hat{D}_q\hat{D}_l\hat{F}_{ij}] - i[\hat{F}_{ij},\frac{1}{2}\{\hat{A}_k,\{\hat{A}_p,\hat{F}_{lq}\}\} + \frac{1}{2}\{\hat{A}_p,\{\hat{A}_k,\hat{F}_{lq}\}\} + \\
& + \frac{1}{2}[[\hat{A}_k,\hat{A}_p],\partial_l\hat{A}_q + \partial_q\hat{A}_l] - i[\partial_p\hat{A}_k,\partial_l\hat{A}_q] + i[\partial_k\hat{A}_p,\partial_q\hat{A}_l]]\Big) + \\
& + (\alpha,\beta \text{ dependent terms})\,.
\end{aligned} \tag{5.1}$$

$$
\begin{aligned}
[\delta_1, \delta_2]\hat{A}_i = & \frac{1}{16}\delta\theta_2^{kl}\delta\theta_1^{pq}\Big(2i[\hat{F}_{kp}, \hat{D}_l\hat{F}_{qi} + \hat{D}_q\hat{F}_{li}] + \\
& +\hat{D}_i\Big(\frac{1}{2}\{\hat{A}_k, \{\hat{A}_p, \hat{F}_{lq}\}\} + \frac{1}{2}\{\hat{A}_p, \{\hat{A}_k, \hat{F}_{lq}\}\} + \\
& +\frac{1}{2}[[\hat{A}_k, \hat{A}_p], \partial_l\hat{A}_q + \partial_q\hat{A}_l] - i[\partial_p\hat{A}_k, \partial_l\hat{A}_q] + i[\partial_k\hat{A}_p, \partial_q\hat{A}_l]\Big)\Big) + \\
& +\hat{D}_i(\alpha, \beta \text{ dependent terms}).
\end{aligned} \tag{5.2}
$$

The first term on the right hand side is 'local' and gauge-covariant. So, it can be absorbed by a *field redefinition* of the gauge field $\hat{A}_i$.[8] The rest is of the form of gauge transformation. The former corresponds to the ambiguity of $\hat{G}_i^{(n)F}$ and the latter to that of $\hat{G}_i^{(n)}$ in (3.8).

This shows that, if we require a 'physical input' such that a noncommutative gauge field would be defined only up to *field redefinitions*, physics does not generally depend on paths in θ-space.[9]

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[8] This was suggested by Y. Okawa and E. Witten.

[9] Of course, by a choice of a path (or a field redefinition), the functional form of the action changes.